Multi-tape Computing with Synchronous Relations

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There are many motivations for using multi-tape transducers:

- We want to relate more than two aspects of a language: e.g. semantics, morphology, phonology, phonetics.
- We want to keep track of intermediate steps in composition of relations: e.g. in old language reconstruction.
- We want to relate more than two languages.
- ...
However, multi-ary relations are not usually supported by standard libraries, and behave differently from binary relations in some ways.

**Our solution**

Our solution is to encode multi-ary relations as binary/unary relations.

However, in general, we cannot encode arbitrary rational (transducer recognizable) relations as unary relations (see below). But this is possible with the **synchronous rational relations (SR)**.
Multi-tape Computing with Synchronous Relations

Synchronous rational relations are in a sense a largest subclass of the rational relation, which forms a Boolean algebra. Hence:

- Closure under intersection, complement (contrary to rational relations)
- Consequently: decidability inclusion and equivalence of two relations (contrary to rational relations)
- Closure under generalized (lossless) composition, i.e. matching of one or more components, with and without cancelling out.
Problem: Synchronous rational relations are inconvenient to use for the community:

- Rational expressions (as in FOMA [Hulden, 2009]) do not allow to characterize SR.
- Solution: we describe a class of expressions which exactly characterizes SR.
Problem: Synchronous rational relations are inconvenient to use for the community:

- (Synchronous) multi-tape relations are not supported by standard libraries
- Solution: we implement an interface which translates SR-expressions to regular languages, faithfully encoding all operations. These can be handled by standard libraries.
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Problems of rational relations

Synchronous rational relations

The encoding: synchronous factorizations

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Problems of rational relations

Rational relations
A relation is rational if it is denoted by some rational expression
Fix an arbitrary alphabet $\Sigma$ and an arbitrary arity $n$

- for $a_1, \ldots, a_n \in \Sigma \cup \{\epsilon\}$, $(a_1, \ldots, a_n)$ is a rational expression (denoting $\{(a_1, \ldots, a_n)\}$)
- if $e, f$ are rational expressions, then so is $e \cdot f$ (denoting componentwise concatenation of tuples),
- if $e, f$ are rational expressions, then so is $e + f$ (denoting union)
- if $e$ is a rational expressions, then so is $e^*$ (denoting $1 + e + (e \cdot e) + \ldots$, where $1 = \{(\epsilon, \ldots, \epsilon)\}$
Problems of rational relations

Rational (transducer recognizable) relations are extremely useful in NLP. This is based on a number of properties:

- Closure under composition
- Closure under union
- Closure under concatenation and Kleene star

Each of these operations is very useful, because it allows to construct a complex relation by simpler ones by means of the operations. Closure ensures we still have finite-state transducers effectively computing the relation.
Problems of rational relations

Problems
Rational relations are not closed under intersection (for proof, see [Berstel, 1979]), and consequently not under complement.

- libraries as FOMA have a pseudo-intersection operation, but it is not guaranteed to yield a mathematically correct result
- without intersection and complement, we cannot decide whether two transducers compute the same relation.
- all existing libraries for transducers and rational relations only support binary relations
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Convolution (tuple of strings)

Put $\Sigma_\bot := \Sigma \cup \{\bot\}$, for $\bot \notin \Sigma$.

The convolution of a tuple of strings $(w_1, \ldots, w_i) \in (\Sigma^*)^i$, written as

$$\otimes(w_1, \ldots, w_i),$$

is in $((\Sigma_\bot^*)^i$ and of length $\max(\{|w_j| : 1 \leq j \leq i\})$, defined as follows: the $k$th letter-tuple of $\otimes(w_1, \ldots, w_i)$ is $\langle \sigma_1, \ldots, \sigma_i \rangle$, where $\sigma_j$ is the $k$-th letter of $w_j$ provided that $k \leq |w_j|$, and $\bot$ otherwise.
Synchronous rational relations

Convolution (relation)

The convolution of a relation \( R \subseteq (\Sigma^*)^i \) is defined by
\[
\otimes R := \{ \otimes(w_1, \ldots, w_i) : (w_1, \ldots, w_i) \in R \}.
\]

Synchronous regular relations

A relation \( R \in (\Sigma^*)^i \) is synchronous regular, if there is an \( \epsilon \)-free finite-state automaton with transitions labelled by \((\Sigma_{\perp})^i \) recognizing \( \otimes R \).

Example

\((a, a) \cdot (a, \epsilon))^* \not\in \text{SR} \), because no \( \epsilon \)-free transducer recognizes
\[
\{(a^{2n}, a^n\perp^n) : n \in \mathbb{N}_0 \}
\]
Why Synchronous rational relations?

Largest natural subclass
The class \( SR \) is the largest known natural class smaller than the rational relations.

Advantages of \( SR \)

- \( SR \) has a number of important closure properties: composition, projection, cylindrification (see below)
- In particular, \( SR \) is a Boolean algebra, hence inclusion and equivalence are decidable!
- But: \( SR \) is not closed under concatenation and Kleene star!
- We will use the fact there is an interesting correlation between \( SR \) and the regular languages.
Synchronous rational relations: operations

Projection

We define for $i \leq n$, $R \subseteq (\Sigma^*)^n$,

$$\pi_i(R) = \{(w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n): (w_1, \ldots, w_n) \in R\}$$

Cylindrification

For $i \leq n + 1$, $R \subseteq (\Sigma^*)^n$,

$$C_i(R) = \{(w_1, \ldots, w_{i-1}, v, w_i, \ldots, w_n): v \in \Sigma^*, (w_1, \ldots, w_n) \in R\}$$

Homomorphisms

$h : (\Sigma^*)^n \rightarrow (T^*)^n$ is a homomorphism, if

$h(w_1, \ldots, w_n) = (h(w_1), \ldots, h(w_n))$, and $h(aw) = h(a)h(w)$.

$h$ is a relabelling, if $a \in \Sigma$ implies $h(a) \in T$. 
Synchronous rational relations: operations

Composition and generalized composition

These operations are not among the standard finite-state operations. But: together with Boolean operations, they allow to define

- Relation composition \(((a, b) \circ (b, c) \mapsto (a, c))\)
- Lossless relation composition \(((a, b) \oplus (b, c) \mapsto (a, b, c))\)
- Generalized composition of relations of higher arity (matching more than one component, e.g. \((a, b, c) \circ_2 (b, c, d) \mapsto (a, d))\)
- Same for lossless composition e.g. \((a, b, c) \oplus_2 (b, c, d) \mapsto (a, b, c, d))\)
Synchronous rational relations

Closure properties of SR

1. Boolean closure: If \( R, S \subseteq (\Sigma^*)^n \), \( R, S \in \text{SR} \), then \((\Sigma^*)^n - R, S \cup R, S \cap R \in \text{SR}\).

2. Projection/Cylindrification: If \( R \subseteq (\Sigma^*)^n \), \( R \in \text{SR} \), then \( \pi_i(R), C_i(R) \in \text{SR} \).

3. Generalized (lossless) composition: If \( R \subseteq (\Sigma^*)^m \), \( S \subseteq (\Sigma^*)^n \), \( o \leq m, n \), then if \( R, S \in \text{SR} \), then \( R \circ_o S, R \oplus_o S \in \text{SR} \).

4. Relabelling: If \( R \in \text{SR} \), \( h \) a relabelling, then \( h[R] \in \text{SR} \). If \( h \) a homomorphism, then \( h[L] \in R \) (the rational relations).
Problem: concatenation and star
SR lacks closure under concatenation and Kleene star
if $R, S \subseteq (\Sigma^*)^n$, $R, S \in \text{SR}$, then $R \cdot S$ and $R^*$ need not be in SR.

Example
$(a, a)^*$, $(b, \epsilon)^*$ and $((a, a) \cdot (a, \epsilon))$ are in SR, but
\begin{itemize}
  \item $(b, \epsilon)^* \cdot (a, a)^*$ $\notin$ SR
  \item $((a, a) \cdot (a, \epsilon))^*$ $\notin$ SR
\end{itemize}
Interim summary

What we have showed
These properties allow us to use $\text{SR}$ for multitape computing. However, the main problem remains: standard libraries do not support more than binary relations.

How we proceed
We will tackle this problem by encoding arbitrary synchronous rational relations as regular languages.
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The encoding: synchronous factorizations

The implementation
The encoding: synchronous factorizations

We say a map $\psi : (T^*)^n \to \Sigma^*$ encodes tuples in strings, if there are maps $\phi_1, ..., \phi_n$ such that for all $i : 1 \leq i \leq n$,

$$\phi_i(\psi(w_1, ..., w_n)) = w_i$$  \hspace{1cm} (1)

Faithfulness
Let $R_1, ..., R_n$ be relations, $\tau$ be an $n$-ary operation, $\psi$ be an encoding. Then we say that the operation $\tau_\psi$ faithfully encodes $\tau$, if

$$\psi(\tau(R_1, ..., R_n)) = \tau_\psi(\psi(R_1), ..., \psi(R_n))$$ \hspace{1cm} (2)

This states that we can simulate operations on relations via operations on their code.
The encoding: synchronous factorizations

Our encoding
It is based on tuple concatenation, but not componentwise: we defined $\cdot$ by

$$(a, b) \cdot (c, d) = (ac, bd),$$

which results in a *tuple of strings*. To encode tuples as strings, we form

$$(a, b)(c, d) \text{ (without $\cdot$)},$$

which is not a tuple of strings, but rather a *string of tuples*. 
The encoding: synchronous factorizations

Factorization
We say that a string of tuples $\vec{x}_1...\vec{x}_i$ is a factorization of $\vec{y} \in (\Sigma^*)^n$, if

1. $\vec{x}_1, ..., \vec{x}_i \in (\Sigma \cup \epsilon)^n$, and
2. $\vec{x}_1 \cdot ... \cdot \vec{x}_i = \vec{y}$.

Factorizations are not unique, consider factorizations as $(a, \epsilon)(\epsilon, b)$ of $(a, b)$.

Synchronous factorization
A factorization $\vec{x}_1....\vec{x}_n$ is synchronous, if the following holds: if the $j$th letter of $\vec{x}_i$ is $\epsilon$, then for all $m : i \leq m \leq n$, the $j$th letter of $\vec{x}_m$ is $\epsilon$.

The synchronous factorization of a tuple is unique, hence we have a function $synfact(\vec{x})$
The encoding: synchronous factorizations

We generally put $f[X] := \{f(x) : x \in X\}$

**Lemma**
Assume $R \subseteq (\Sigma^*)^n$. Then $R$ is synchronous rational if and only if $\text{synfact}[R]$ is a regular language.

**Example**
\[
\text{synfact}[(a, a) \cdot (a, \epsilon)] = \{ (a, a)^n (a, \epsilon)^n : n \in \mathbb{N}_0 \},
\]
which is obviously not regular (isomorphic to $a^n b^n$).
The encoding: synchronous factorizations

The previous lemma shows the tight relation between $\text{SR}$ (of arbitrary arity) and the regular languages. For the rational relations, we can show that no such encoding exists:

Lemma

There is no rational (i.e. finite-state computable) encoding

$$\psi : (\Sigma^*)^n \rightarrow T^*$$

such that for all rational relations $R$, $\psi[R]$ is regular.

This is the main motivation for using $\text{SR}$!
The encoding: synchronous factorizations

Here some faithful encodings of standard operations, given the encoding via synchronous factorizations.

Standard operations

\( \tau \) (on relation) \( \tau_{\psi} \) (on language)

1. \( \psi(R \cup S) \) \( \psi(R) \cup \psi(S) \)
2. \( \psi(R \cap S) \) \( \psi(R) \cap \psi(S) \)
3. \( \psi(\overline{R}) \) \( \psi(\overline{R}) \cap \text{code}_{\psi} \)
4. \( \psi(\pi_i(R)) \) \( h_i[\psi(R)], h_i \text{ a relabelling} \)
5. \( \psi(C_i(R)) \) \( h_i^{-1}(\psi(R)), h_i \text{ a relabelling} \)
6. \( \psi(R \circ_1 S) \) \( \pi_2(C_3(\psi(R)) \cap C_1(\psi(S))) \)
7. \( \psi(R \oplus_1 S) \) \( C_3(\psi(R)) \cap C_1(\psi(S)) \)
8. \( R \circ_i S \) generalize 6.
9. \( R \oplus_i S \) generalize 7.
10. \( \psi(R^{-1}) \) \( h[\psi(R)], h \text{ a relabelling} \).
The encoding: synchronous factorizations

Problem
Our encoding is not faithful for concatenation and Kleene star. This follows from two facts:

Lemma
*If we close the class of synchronous rational relations under concatenation and Kleene star, we obtain the rational relations.*

And:

Lemma
*There is no rational encoding \( \psi : (\Sigma^*)^n \rightarrow T^* \) such that for all rational relations \( R \), \( \psi[R] \) is regular.*
The encoding: synchronous expressions

Still, we want to use concatenation and Kleene star in a restricted ("safe") fashion!

▶ Therefore, we devise a category system for expressions with \( \cdot \) and \( * \).
▶ Categories tell us, whether an expression can still be guaranteed to denote a synchronous relation, and
▶ for every synchronous rational relation, we have an expression of a "safe" category!

Note however that in general, it is undecidable whether a rational expression denotes a relation in \( \text{SR} \)! 
We distinguish these categories of rational expressions:

1. \textit{el}, the equal-length expressions (all components have equal length, e.g. \((a, b, c)^*)\)

2. \textit{ed}, the \(\epsilon\)-difference expressions, where shorter components are \(\epsilon\) (e.g. \((a, \epsilon, c)^*)\)

3. \textit{bd}, the bounded length-difference expressions (e.g. \((a, b)^* \cdot (a, \epsilon)\))

4. \textit{gd}, where difference can be unbounded and shorter components need not be \(\epsilon\), (e.g. \(((a, a)^* \cdot (b, \epsilon)^*))\)

5. \(\bot\), the expressions which are no longer guaranteed to be synchronous
We call the expressions of category $el, bd, ed, gd$ the **synchronous rational expressions** (SR-expressions); this consequently forms a (proper) subset of the rational expressions. If we extend these expressions with constructors for projection, cylindrification and Boolean operations, we obtain the following:

**Lemma**

*Soundness* Every extended synchronous rational expression denotes a synchronous rational relation.

**Lemma**

*Completeness* For every synchronous rational relation, there is an extended synchronous rational expression denoting it.
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The goal of our implementation is to be able to process multi-ary relations with a standard library, in a transparent way.

user $\iff$ interface $\iff$ existing FS-library

- We do not implement the standard operations, but use the ones of the library
- The language used for the input is as close as possible to the one of the library
- The input is type-checked, and encoded to be processed (or not) by the library
The implementation: example (type-checking)

The following input has to be ruled out by the type-checker:

\[(a, \varepsilon, b)^* (a, c, a) \mid (a, c, b)^* \]

\[(a, \varepsilon, b)^* (a, c, a) \]
The implementation: example (type-checking)

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The implementation: example (type-checking)

The following input has to be ruled out by the type-checker:

\[
( (a, \text{epsilon}, b) \ast (a, c, a) ) \mid (a, c, b) \ast
\]

\[
(a, \varepsilon, b) \ast (a, c, a)
\]

The expression does not belong to the class, the process is stopped
The implementation: example (encoding)

For the following input, the type-checking is successful and the encoding can be given to the library

\[
| ( (a, \varepsilon, b) (a, c, a) ) | (a,c,b)^* 
\]
The implementation: example (encoding)

For the following input, the type-checking is successful and the encoding can be given to the library

\[ (a, \text{epsilon}, b) (a, c, a) \mid (a, c, b) \ast \]

\[ [ 'concat', [ ( 'a', 'epsilon', 'b' ) ], [ ( 'a', 'c', 'a' ) ] ] \]

should be forbidden, but an expression denoting the same language can be obtained by \( \varepsilon \)-shifting
The implementation: example (encoding)

For the following input, the type-checking is successful and the encoding can be given to the library

\[(a, \text{epsilon}, b) \ (a, c, a) \] | \((a,c,b)^*\)

\[['concat', [( 'a', 'epsilon', 'b' )],[( 'a', 'c', 'a' )]]\]

should be forbidden, but an expression denoting the same language can be obtained by \(\varepsilon\)-shifting

\[['union',[['concat',[( 'a', 'c', 'b' )],[( 'a', 'epsilon', 'a' )]],[['star',[( 'a', 'c', 'b' )]]]]\]
The implementation: example (encoding)

For the following input, the type-checking is successful and the encoding can be given to the library

\[(a, \text{epsilon}, b) (a, c, a) \mid (a, c, b)^*\]

\[\text{'concat'}, [(\text{'}a\text{'}, \text{'}epsilon\text{'}, \text{'}b\text{'})],[(\text{'}a\text{'}, \text{'}c\text{'}, \text{'}a\text{'}))] \]

should be forbidden, but an expression denoting the same language can be obtained by \(\varepsilon\)-shifting

\[\text{'union'}, [\text{'concat'}, [(\text{'}a\text{'}, \text{'}c\text{'}, \text{'}b\text{'})],[(\text{'}a\text{'}, \text{'}epsilon\text{'}, \text{'}a\text{'})]],
\quad [\text{'star'},[(\text{'}a\text{'}, \text{'}c\text{'}, \text{'}b\text{'})]] \]

\[\left(\%[\text{'a'}\%,\text{'c'}\%,\text{'b'}\%] \%[\text{'a'}\%,\text{epsilon} \%,'a'\%]\right) \mid \left(\%[\text{'a'}\%,\text{'c'}\%,\text{'b'}\%\right)^*\]
Conclusion

- Idea: allow users to work with (synchronous) rational relations of arbitrary arity.
- Method: encoding arbitrary relations as simple languages to work with existing libraries.
- Problem: cannot work with the full class of rational relations.
- Synchronous rational relations: only concatenation and star are problematic.
- We presented a class of expressions which denotes all and only the synchronous rational expressions.
- Practical side: type checker for expressions and implementation of faithful encoding of operations.
Thank you!